Lecture Notes, Lecture 16

7.1 Existence of Equilibrium

$$P = \left\{ p \middle| p \in R^N,\ p_k \geq 0,\ k = 1\ldots, N,\ \sum_{k=1}^{N} p_k = 1 \right\}$$

$$\tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - r$$

$$= \sum_{i \in H} x^i - \sum_{j \in F} y^j - r$$

, where x^i is household i's consumption plan, y^j is firm j's production plan and r is the resource endowment of the economy. $\tilde{Z}(p)$ is the economy's excess demand function. Recall that all of the expressions in $\tilde{Z}(p)$ are N-dimensional vectors.

Definition: $p^0 \in P$ is said to be an equilibrium price vector if $\tilde{Z}(p^0) \leq 0$ (the inequality holds co-ordinatewise) with $p^o_k = 0$ for k such that $\tilde{Z}_k(p^0) < 0$. That is, p^0 is an equilibrium price vector if demand equals supply except for free goods, $\sum_{i \in H} \tilde{D}^i(p^0) \leq \sum_{j \in F} \tilde{S}^j(p^0) - r$.

Weak Walras' Law (Theorem 6.2): For all $p \in P,\ p \cdot \tilde{Z}(p) \leq 0$. For p such that $p \bullet \tilde{Z}(p) < 0$, there is k = 1, 2, ..., N, so that $\tilde{Z}_k(p) > 0$, assuming C.I - C.V, C.VII, C.VIII.

Continuity: $\tilde{Z}(p)$ is a continuous function, assuming P.II, P.III, P.V,P.VI and C.I-C.V, C.VII-C.VIII (Theorem 4.1, Theorem 5.2, Theorem 6.1).

Theorem 7.1: Assume P.II, P.III, P.V, P.VI, and C.I-C.V, CVII-C.VIII. There is $p^* \in P$ so that p^* is an equilibrium.

Proof: $T : P \to P$. For each k= 1,2,3, ..., N.

$$T_k(p) \equiv \frac{p_k + \max\left[0, \tilde{Z}_k(p)\right]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p)\right]} = \frac{p_k + \max\left[0, \tilde{Z}_k(p)\right]}{\sum_{n=1}^{N} \{p_n + \max\left[0, \tilde{Z}_n(p)\right]\}}.$$

By the Brouwer fixed point theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all k = 1, ..., N,

$$T_k(p_k^*) = p_k^* = \frac{p_k^* + \max\left[0, \tilde{Z}_k(p^*)\right]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]}$$

Thus, either $p_k^* = 0$ or

$$p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} > 0 \quad .$$

Case 1: $p_k^* = 0 = \max\left[0, \tilde{Z}_k(p^*)\right]$. Hence $\tilde{Z}_k(p^*) \le 0$.

Case 2: $p_k^* = \dfrac{p_k^* + m\mathrm{ax}\left[0, \tilde{Z}_k(p^*)\right]}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} > 0$

To avoid repeated tedious notation, let

$$0 < \alpha = \frac{1}{1 + \sum_{n=1}^{N} \max\left[0, \tilde{Z}_n(p^*)\right]} \le 1 \quad .$$

We have

$$p_k^* = \alpha p_k^* + \alpha \max[0, \tilde{Z}_k(p^*)]$$

$$(1-\alpha)p_k^* = \alpha \max[0, \tilde{Z}_k(p^*)]$$

Multiplying through by $\tilde{Z}_k(p^*)$,

$$(*) \quad (1-\alpha)p_k^* \tilde{Z}_k(p^*) = \alpha(\max[0, \tilde{Z}_k(p^*)])\tilde{Z}_k(p^*)$$

Restating the Weak Walras' Law,

$$0 \geq p^* \cdot \tilde{Z}(p^*) = \sum_{k \in Case\ 1} p_k^* \tilde{Z}_k(p^*) + \sum_{k \in Case\ 2} p_k^* \tilde{Z}_k(p^*)$$

$$= 0 + \sum_{k \in Case\ 2} p_k^* \tilde{Z}_k(p^*) = \sum_{k \in Case\ 2} p_k^* \tilde{Z}_k(p^*)$$

or

$$0 \geq \sum_{k \in Case\ 2} p_k^* \tilde{Z}_k(p^*)$$

Multiplying through by (1-α), and substituting (*) we have

$$0 \geq (1-\alpha) \sum_{k \in Case\ 2} p_k^* \tilde{Z}_k(p^*)$$
$$= \alpha \sum_{k \in Case\ 2} (\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*).$$

But this means that $\tilde{Z}_k(p^*) \leq 0$, for all k in case 2.

But then, there is no k, either in case 1 or 2, so that $\tilde{Z}_k(p^*)$ >0. But the Weak Walras' Law says that if $p^* \cdot \tilde{Z}(p^*) < 0$, it follows that there is k so that $\tilde{Z}_k(p^*) > 0$. Hence we must have $p^* \bullet \tilde{Z}(p^*) = 0$. Thus for k so that $\tilde{Z}_k(p^*) < 0$, it follows that $p_k^* = 0$. This completes the proof.

Q.E.D.

Theorem 7.1 is a proof of the consistency of the competitive model of chapters 4-7. It is possible to find prices, $p^* \in P$ so that competitive markets clear. When economists talk about competitive market prices finding their own level, they are not necessarily speaking vacuously. Under the hypotheses above, there is a competitive equilibrium price system.

Lemma 7.1: Assume P.II, P.III, P.V, P.VI, and C.I-C.V, CVII-C.VIII. Let p* be an equilibrium. Then $|\widetilde{D}^i(p^*)| < c$ where c is the bound on the Euclidean length of demand, $\widetilde{D}^i(p)$. Further, in equilibrium, Walras' Law holds as an equality, $p^* \cdot \widetilde{Z}(p^*) = 0$.

Proof: Since $\widetilde{Z}(p^*) \leq 0$ (co-ordinatewise), we know that $\sum_{i \in H} \widetilde{D}^i(p^*) \leq \sum_{j \in F} \widetilde{S}^j(p^*) + \sum_{i \in H} r^i$, co-ordinatewise. But that implies that the aggregate consumption $\sum_{i \in H} \widetilde{D}^i(p^*)$ is attainable, so for each household i, $|\widetilde{D}^i(p^*)| < c$ where c is the bound on demand, $\widetilde{D}^i(p)$.

We have for all p, $p \cdot \widetilde{Z}(p) \leq 0$. In equilibrium, at p*, we have $\widetilde{Z}(p^*) \leq 0$ with $p^*_k = 0$ for k so that $\widetilde{Z}_k(p^*) < 0$. Therefore $p^* \cdot \widetilde{Z}(p^*) = 0$. QED